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1992 J. Phys. A: Math. Gen. 25 3567

(http://iopscience.iop.org/0305-4470/25/12/018)

J. Phys. A: Math. Gen. 25 (1992) 3567–3592. Printed in the UK

On the scaling laws of higher-order q-phase transitions

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Received 19 February 1992, in final form 2 March 1992

Abstract. q-phase transition points in the context of the thermodynamical formalism of dynamical systems arise via the degeneracy of eigenvalues of the corresponding transfer operator. The scaling behaviour near bifurcation points of dynamical systems is investigated by a mean-field-like expansion for the characteristic equation of this operator. Scaling relations in the vicinity of q-phase-transition points, which are brought about by a doubly (respectively triply) degenerated eigenvalue, are explicitly derived. For the characteristic function (topological pressure) this relation reads $\phi(q) \simeq \ln \nu_* + \delta^a \tilde{\phi}((q-q_*)/\delta^a)$ where the exponent $a = 1, \frac{1}{2}, \frac{1}{3}$ of the bifurcation parameter δ depends on general properties of the phase-transition point. The approach explains the universal features of the scaling behaviour.

1. Introduction

The thermodynamical formalism originally introduced in the context of the ergodic theory of dynamical systems and the mathematical formulation of equilibrium statistical mechanics [1, 2] has been applied recently to problems in nonlinear dynamics, chaotic systems and turbulence [3–5]. The aim of this approach consists in the investigation of temporal coarse grained, that means finite-time averaged, quantities and their fluctuations due to the irregular motion in the system under consideration. Usually one considers the fluctuations of the local expansion rate, which is due to Bowen's theorem [2], of special importance; but also different quantities have been treated in this context [5]. It has been shown that the large fluctuations contain the essential information about the dynamics and the structure of the strange invariant set [6]. These large fluctuations can be described appropriately by a characteristic function $\phi(q)$, referred to as topological pressure in the mathematical literature, which corresponds to the free energy of statistical mechanics in the previously mentioned thermodynamical formulation. Non-analyticities in this function, called q-phase transitions, indicate a singular local structure of the chaotic attractor [7]. For this reason these transitions can be observed at bifurcation points of dynamical systems, especially at crisis points, and a typical scaling behaviour in the quantities of interest emerges in their vicinity [8–10]. In a preceding publication we have pointed to an explanation for the surprising fact that the corresponding scaling functions do not depend on the special system under consideration [11]. It is the aim of this article to extend this idea to more complicated bifurcations.

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To be definite and to state the notation let us consider a discrete dynamical system $x_{n+1} = T(x_n)$ although our approach is applicable to more general situations as will become clear in the sequel. As a slight generalization of the usual thermodynamical formalism we want to investigate the fluctuations of several scalar quantities $u_1(x), \ldots, u_M(x)$ [12]. Following the lines of the one observable case the essential information concerning the nonlinear system is contained in the generating function

$$\left\langle \exp\left(q\sum_{i=0}^{n-1} u(T^i(x))\right)\right\rangle = (\nu_q^{(0)})^n \left\{ J_q^{(0)} + \sum_{l>0} J_q^{(l)} \left(\nu_q^{(l)}/\nu_q^{(0)}\right)^n \right\}. \tag{1}$$

Here $q = (q_1, \ldots, q_M)$ and $u = (u_1, \ldots, u_M)$ denote a shorthand vector notation, T^i means the i-times iterated map T and the ensemble average $\langle \ldots \rangle$ is meant with respect to a distribution of initial points which is usually assumed to be the natural one (SRB measure). The expansion on the right-hand side of (1) can be understood easily by using a transfer operator whose explicit expression reads in this context as [13]

$$(\mathcal{H}_q^u h)(x) := \int \delta(x - T(y)) e^{q \cdot u(y)} h(y) dy. \tag{2}$$

The main behaviour of the expansion (1) is determined by the eigenvalues $\nu_q^{(l)}$ of this operator† where for simplicity in the notation we want to assume a discrete spectrum ordered according to the relation $\nu_q^{(0)} \geqslant |\nu_q^{(l)}| \geqslant |\nu_q^{(k)}|$, $0 < l \leqslant k$ although a continuous part can be incorporated directly in our approach. The quantities of interest, the characteristic function (topological pressure)

$$\phi(q) = \lim_{n\to\infty} \frac{1}{n} \ln \left\langle \exp\left(q \sum_{i=0}^{n-1} u(T^i(x))\right)\right\rangle = \ln \nu_q^{(0)} \tag{3}$$

which determines the stationary fluctuations of $u(x)$ as soon as the damping rates $\gamma_q^{(l)}$ and the corresponding frequencies $\omega_q^{(l)}$ governing the temporal correlations [14]

$$\gamma_q^{(l)} + i\omega_q^{(l)} := -\ln \frac{\nu_q^{(l)}}{\nu_q^{(0)}} \tag{4}$$

can be related to the eigenvalues of the operator (2). A phase transition that means a non-analyticity in the quantities (3) and (4) is brought about by a degeneracy of eigenvalues. This situation can occur at bifurcation points of dynamical systems [9, 15]. In a preceding publication [11] we have shown that, for the case of a doubly degenerated eigenvalue, the scaling behaviour in the vicinity of a bifurcation point can be obtained in a general way from the characteristic equation of the operator (2)

$$P(\nu_q, \delta, q) = 0. \tag{5}$$

Here $\delta \geqslant 0$ denotes a bifurcation parameter leading to a bifurcation that means leading to a degeneracy of eigenvalues in the limit $\delta \downarrow 0$. It is the objective of this

† $J_q^{(l)}$ denote some expansion coefficients.

publication to extend this approach beyond the case of a doubly degenerated eigenvalue as well as to incorporate the multivariable case $q = (q_1, \ldots, q_M)$. It should be pointed out that our approach yields a theoretical explanation for the occurrence of the scaling behaviour and the general form of the scaling functions which is independent of the model under consideration. In section 2 we review the previously treated situation for the broader multivariable case. Section 3 contains the discussion of the more involved case of a triply degenerated eigenvalue. Some illustrative examples are investigated in section 4. Finally our results will be summarized.

2. Doubly degenerated eigenvalue

The main idea in deriving the scaling relation from (5) is based on a few reasonable presuppositions and needs no explicit reference to a special dynamical system. For the situation treated in this section these presuppositions read as follows.

(P_d1) Equation (5) should yield two largest eigenvalues which are well separated from the remaining part of the spectrum. The largest eigenvalue should be real. In the limit $\delta \downarrow 0$ they should become degenerated $\nu_q^{(0)}(\delta \downarrow 0) = \nu_q^{(1)}(\delta \downarrow 0)$ for certain q values leading to a phase transition.

(P_d2) Equation (5) should be analytic in δ and q.

(P_d3) The system should admit an attracting set so that $q = \mathbf{0}$, $\nu_q = 1$ is the largest solution of (5) for all values of δ [16]. That means

$$P(\nu_q = 1, \delta, q = \mathbf{0}) = 0. \tag{6}$$

Before we proceed a few remarks on the meaning of these presuppositions seem to be suitable. The first presupposition restricts the bifurcation of the dynamical system to a certain class which contains, for example, the symmetry breaking chaos transition [15]. Especially when chaotic sets are involved in the bifurcation it has turned out that often only a finite number of eigenvalues govern the phase transition. The second presupposition puts some constraints on the choice of the bifurcation parameter. Our parameter δ is in general a function of the bifurcation parameters of the original system whose explicit relation to the latter is not needed for the present purpose. Although our approach cannot yield this relation *a priori* it can be easily determined *a posteriori* if the scaling behaviour is, for example, computed numerically. Contrary to these the third presupposition is not essential and can be omitted if one wants to treat repelling invariant sets.

Let us begin the derivation of the scaling relations by inspecting the situation at the bifurcation point $\delta \downarrow 0$. In general the eigenvalues $\nu_q^{(l)}(\delta \downarrow 0)$ can be viewed as hyperspheres in the q–ν_q-space where due to (P_d1) the hyperspheres $\nu_q^{(0)}(\delta \downarrow 0)$ and $\nu_q^{(1)}(\delta \downarrow 0)$ cross along the set (cf figure 1)

$$\Gamma_* := \{q | \nu_q^{(0)}(\delta \downarrow 0) = \nu_q^{(1)}(\delta \downarrow 0)\}. \tag{7}$$

This codimension 1 manifold in the q-space yields the phase-transition line. Figure 1 displays the geometrical settings described earlier. Let us now consider how this situation changes if a non-vanishing bifurcation parameter is introduced. For this reason let $q_* \in \Gamma_*$ denote a fixed but arbitrary chosen point on the phase-transition

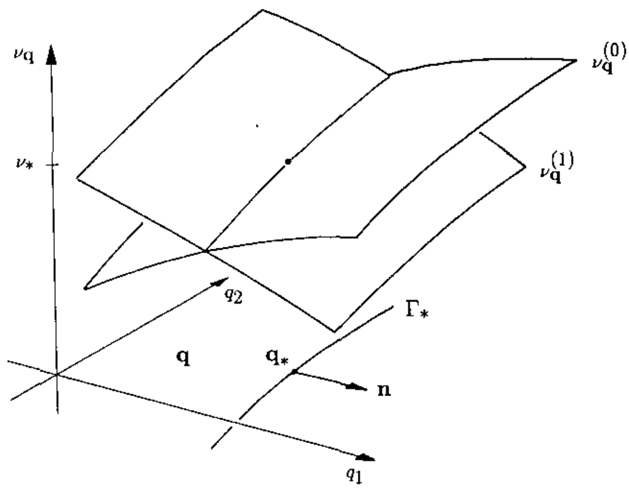

Figure 1. Diagrammatic view of the eigenvalues governing a phase transition based on a double degeneracy ($\delta \downarrow 0$). $\nu_q^{(0)}$, $\nu_q^{(1)}$ denote the two largest eigenvalues and Γ_* the phase-transition line in the two-dimensional q-space ($M = 2$). The fixed chosen phase-transition point (ν_*, q_*) and a vector n normal to the phase-transition line are also indicated.

line and $\nu_* := \nu_{q_*}^{(0)}(\delta \downarrow 0)$ the corresponding eigenvalue. Owing to ($P_d 1$) a second-order polynomial can be extracted from the characteristic equation so that (5) reads

$$0 = P(\nu_q, \delta, q) = \left\{ \left(\frac{\nu_q}{\nu_*} - 1 \right)^2 + f(\delta, q) \left(\frac{\nu_q}{\nu_*} - 1 \right) + g(\delta, q) \right\} \bar{P}(\nu_q, \delta, q) \qquad (8)$$

where $\bar{P}$ denotes the non-vanishing part in the vicinity of the phase-transition point $\nu_q = \nu_*$, $\delta \downarrow 0$, $q = q_*$. The first factor is written in the variable $\nu_q/\nu_* - 1$ for simplicity. For $\tilde{q} \in \Gamma_*$, $\delta \downarrow 0$ equation (8) yields, due to ($P_d 1$), a doubly degenerated solution so that

$$g(\delta \downarrow 0, \tilde{q}) = \tfrac{1}{4} f^2(\delta \downarrow 0, \tilde{q}) \qquad \tilde{q} \in \Gamma_*. \qquad (9)$$

Especially for $\tilde{q} = q_*$ this eigenvalue is given by ν_* which results in the stronger relations

$$f(\delta \downarrow 0, q_*) = 0 \qquad g(\delta \downarrow 0, q_*) = 0. \qquad (10)$$

After these general considerations ($P_d 2$) guarantees the existence of a Taylor expansion of the functions f and g. Taking (10) into account it reads

$$\begin{aligned} f(\delta, q) &= f^{10}\delta + f^{01} : \Delta q : + O_2 \\ g(\delta, q) &= g^{10}\delta + g^{01} : \Delta q : + g^{20}\delta^2 + g^{11} : \Delta q : \delta + g^{02} : \Delta q : \Delta q : + O_3 \end{aligned} \qquad (11)$$

where the abbreviating notation

$$\mathbf{B}\underbrace{: x : x : \ldots : x :}_{n-\text{times}} = \sum_{\mu\nu\ldots\omega} B_{\mu\nu\ldots\omega} \underbrace{x_\mu x_\nu \ldots x_\omega}_{n-\text{times}} \qquad (12)$$

and

$$\Delta q := q - q_* \tag{13}$$

has been introduced and O_n denotes the contribution of nth and higher order. The coefficients in the expansion (11) are now restricted by our presuppositions as can be seen in the following way:

(i) $\delta \downarrow 0$. Then, taking the expansion (11) into account, (8) yields

$$0 = \left(\frac{\nu_q}{\nu_*} - 1\right)^2 + g^{01} : \Delta q : + \mathrm{O}\left(|\Delta q|^2, |\Delta q|\left(\frac{\nu_q}{\nu_*} - 1\right)\right). \tag{14}$$

By ($\mathrm{P_d}$1) this equation admits a real solution. But as Δq has no definite sign the relation

$$g^{01} = \mathbf{0} \tag{15}$$

follows.

(ii) $q = q_*$. Now (8) and (11) lead to

$$0 = \left(\frac{\nu_{q_*}}{\nu_*} - 1\right)^2 + g^{10}\delta + \mathrm{O}\left(\delta^2, \delta\left(\frac{\nu_{q_*}}{\nu_*} - 1\right)\right). \tag{16}$$

The same reasoning as above yields

$$g^{10} \leqslant 0 \tag{17}$$

where the difference from (15) comes from the fact that δ can take only non-negative values.

(iii) $q_* = \mathbf{0}$. This case can occur if the phase-transition manifold (7) crosses the origin of the q-space. Then by ($\mathrm{P_d}$3) $\nu_* = 1$ holds and by making reference to (6), (8) yields,

$$g(\delta, q = \mathbf{0}) = 0 \qquad (q_* = \mathbf{0}). \tag{18}$$

Inspecting the expansion (11) one obtains

$$g^{10} = 0 \qquad g^{20} = 0 \qquad (q_* = \mathbf{0}). \tag{19}$$

Therefore the relation $q_* = \mathbf{0}$ implies $g^{10} = 0$. The reverse is, in general, not valid but holds in the generic case as $q_* \neq \mathbf{0}$, $g^{10} = 0$ would require an additional constraint which can be removed by a small perturbation. For this reason we will concentrate on the cases $q_* = \mathbf{0}$, $g^{10} = 0$ respectively $q_* \neq \mathbf{0}$, $g^{10} < 0$ and omit the previously mentioned non-generic situation.

(iv) $q = \tilde{q} \in \Gamma_*, \delta \downarrow 0$: Then (9) and expansion (11) yield

$$g^{02} : \tilde{q} - q_* : \tilde{q} - q_* : -\tfrac{1}{4}\left(f^{01} : \tilde{q} - q_* :\right)^2 + \mathrm{O}(|\tilde{q} - q_*|^3) = 0 \tag{20}$$

which means that

$$g^{02} : x : x : -\tfrac{1}{4}\left(f^{01} : x :\right)^2 = 0 \qquad \text{if } x \text{ is tangent to } \Gamma_*. \tag{21}$$

Due to this relation the expansion coefficients are related to the shape of the phase-transition line.

With these settings the two different cases $q_* = 0$ and $q_* \neq 0$ can be analysed easily.

Case A. $q_* = 0$. By making reference to (15) and (19) expansion (11) reads

$$f(\delta, q) = f^{10}\delta + f^{01} : q : + O_2$$
$$g(\delta, q) = g^{11} : q : \delta + g^{02} : q : q : + O_3. \tag{22}$$

Using the scaling abbreviations

$$\nu_q - 1 =: \delta\psi_s \qquad q =: \delta s \tag{23}$$

where ψ_s and s are of the order $O(1)$ in the limit of small δ the characteristic equation (8) reads

$$0 = \psi_s^2 + (f^{10} + f^{01} : s :)\psi_s + g^{11} : s : + g^{02} : s : s : + O(\delta) \tag{24}$$

and leads to the following asymptotic behaviour in the region $0 < \delta \ll 1$†

$$\psi_s^{(0/1)} \simeq -\tfrac{1}{2}(f^{10} + f^{01} : s :) \pm \sqrt{\tfrac{1}{4}(f^{10} + f^{01} : s :)^2 - g^{11} : s : - g^{02} : s : s :}. \tag{25}$$

This expression can be considerably simplified if one splits s in a tangent and normal part according to $s = s^{\mathrm{T}} + (sn)n$. Here n denotes a vector normal to the phase-transition manifold at q_* (cf figure 1). Inserting this in the radicant of (25) and using the relation (21) the second-order contribution in s^{T} vanishes. Then the radicant has the general form $A : s^{\mathrm{T}} : + B$ where the coefficients A and B depend on the normal part (sn). If the coefficient A does not vanish identically this expression takes negative values by choosing s^{T} appropriately. As a consequence ψ_s becomes a complex quantity in contradiction to ($\mathrm{P_d}$1). Hence A vanishes identically which means that only the normal part $(sn)n$ contributes to the square root. Finally rewriting (25) for the original quantities (23) one obtains the scaling relation

$$\nu_q^{(0/1)} - 1 \simeq \delta H_{\mathrm{A}}^{(0/1)}(q/\delta)$$
$$H_{\mathrm{A}}^{(0/1)}(x) := -\tfrac{1}{2}(f^{10} + f^{01}x) \pm \sqrt{\tfrac{1}{4}(f^{10} + f^{01}(xn))^2 - g^{11}(xn) - g^{02}(xn)^2} \tag{26}$$

where $f^{01} := f^{01} : n :$, $g^{11} := g^{11} : n :$ and $g^{02} := g^{02} : n : n :$. Both terms of the scaling function can be easily interpreted. The first contribution arises through the q dependence of the eigenvalues which is also present in the 'unperturbed' ($\delta \downarrow 0$) system. The square root yields a q-dependent scaling normal to the phase-transition manifold. If one varies q along this manifold this term yields only a constant difference between the eigenvalues resulting from the finite value of the bifurcation parameter.

Case B. $q_* \neq 0$. Then by (15) and the remark following (19) the expansion (11) reads

$$f(\delta, q) = f^{01} : \Delta q : + O(\delta, |\Delta q|^2)$$
$$g(\delta, q) = g^{10}\delta + g^{02} : \Delta q : \Delta q : + O(\delta^2, \delta|\Delta q|, |\Delta q|^3). \tag{27}$$

† The $+$ sign corresponds to $\psi_s^{(0)}$ as $\nu_q^{(0)} \geqslant \nu_q^{(1)}$.

Introducing the different scaling abbreviations

$$\frac{\nu_q}{\nu_*} - 1 =: \sqrt{\delta}\psi_s \qquad \Delta q =: \sqrt{\delta} s \tag{28}$$

due to the fact that g contains a first-order contribution the characteristic equation (8) reads

$$0 = \psi_s^2 + f^{01} : s : \psi_s + g^{10} + g^{02} : s : s : + \mathrm{O}(\sqrt{\delta}) \tag{29}$$

and possesses the asymptotic solution

$$\psi_s^{(0/1)} \simeq -\tfrac{1}{2} f^{01} : s : \pm\sqrt{\tfrac{1}{4}(f^{01} : s :)^2 - g^{02} : s : s : - g^{10}}. \tag{30}$$

The same reasoning as presented in case A yields the scaling relation

$$\begin{aligned}&\frac{\nu_q}{\nu_*} - 1 \simeq \sqrt{\delta} H_{\mathrm{B}}^{(0/1)}\left(\frac{\Delta q}{\sqrt{\delta}}\right)\\ &H_{\mathrm{B}}^{(0/1)}(x) := -\tfrac{1}{2} f^{01} : x : \pm\sqrt{\left(\tfrac{1}{4}(f^{01})^2 - g^{02}\right)(xn)^2 - g^{10}}\end{aligned} \tag{31}$$

where $f^{01} := f^{01} : n :$, $g^{02} := g^{02} : n : n :$ and n denotes a vector normal to the phase-transition manifold at q_*. Concerning the discussion of the different contributions of the scaling function we refer to case A.

Table 1. Scaling relations in the case of a doubly degenerated eigenvalue and definition of the scaling functions ($X = A, B$). For the explicit expressions we refer to (26_2) and (31_2).

	$q_* = 0$	$q_* \neq 0$
$\phi(q)$	$\delta\bar{\phi}_{\mathrm{A}}(q/\delta)$	$\ln\nu_* + \sqrt{\delta}\bar{\phi}_{\mathrm{B}}\left(\Delta q/\sqrt{\delta}\right)$
$\gamma_q^{(1)}$	$\delta\bar{\gamma}_{\mathrm{A}}(q/\delta)$	$\sqrt{\delta}\bar{\gamma}_{\mathrm{B}}\left(\Delta q/\sqrt{\delta}\right)$
$\omega_q^{(1)}$	0	0
$\bar{\phi}_X(x) := H_X^{(0)}(x)$		
$\bar{\gamma}_X(x) := H_X^{(0)}(x) - H_X^{(1)}(x)$		

It goes without saying that the scaling relations for the characteristic function ϕ and the damping rates can be immediately obtained from (3), (4), (26) and (31). Table 1 summarizes the main results of this section.

3. Triply degenerated eigenvalue

Let us now concentrate on the main part of this article, the case of a bifurcation leading to a triply degenerated eigenvalue of the transfer operator. Before we are going into the details some general remarks seem to be necessary. It is reasonable to assume that simple bifurcations of chaotic sets governed by one bifurcation parameter lead to a doubly degenerated eigenvalue of the transfer operator. This situation is similar to local codimension one bifurcations in the theory of ordinary differential equations where only one or a pair of complex conjugated eigenvalues crosses the imaginary axis. To produce generically higher-order codimension bifurcations several parameters have to be introduced into the unfolding [17]. This indicates that in the case considered here several parameters must also be introduced, whereas we have two possibilities. On the one hand one can consider the situation of more than one bifurcation parameter. On the other hand one can enlarge the set of fluctuating variables $u(x)$. We follow in this article the second idea which yields in a loose thermodynamical analogy a more than one-dimensional phase space (q-space). It has been demonstrated by analysing several examples that in this case a higher-order phase-transition emerges [12, 18]. We want to investigate the scaling behaviour near these phase-transition points from our general point of view.

Let us now state the presuppositions necessary to derive the scaling behaviour from (5):

($\mathrm{P_t}1$) Equation (5) should admit three largest eigenvalues which are well separated from the remaining part of the spectrum. The largest eigenvalue should be positive. In the limit $\delta \downarrow 0$ the eigenvalues should become degenerated $\nu_q^{(0)}(\delta \downarrow 0) = \nu_q^{(1)}(\delta \downarrow 0) = \nu_q^{(2)}(\delta \downarrow 0)$ for certain q values leading to a phase transition.

($\mathrm{P_t}2$) Identical to ($\mathrm{P_d}2$).

($\mathrm{P_t}3$) Identical to ($\mathrm{P_d}3$).

($\mathrm{P_t}4$) In the limit $\delta \downarrow 0$ the three eigenvalues should be real and analytic in a neighbourhood of the phase-transition point.

Concerning ($\mathrm{P_t}1$)–($\mathrm{P_t}3$) we refer the reader to the remarks made in section 2. By ($\mathrm{P_t}4$) we restrict the discussion to phase-transition points where three phase-transition lines meet (cf figure 2). This situation is mostly shared by concrete examples. We are not sure whether the opposite case, that means one real and two complex conjugated eigenvalues in a neighbourhood of the phase-transition point ($\delta \downarrow 0$), can occur in dynamical systems. We refer the reader to appendix A where the details of this case are briefly outlined.

Again we start our discussion by analysing the situation at the bifurcation point $\delta \downarrow 0$. As the eigenvalues are real in the vicinity of the phase-transition point due to ($\mathrm{P_t}4$) they can be viewed as hyperspheres in the q–ν_q-space which cross along the manifolds (cf figure 2)

$$\Gamma_* := \{q | \nu_q^{(0)}(\delta \downarrow 0) = \nu_q^{(1)}(\delta \downarrow 0) = \nu_q^{(2)}(\delta \downarrow 0)\}$$

$$\tilde{\Gamma}_{(i/j)} := \{q | \nu_q^{(i)}(\delta \downarrow 0) = \nu_q^{(j)}(\delta \downarrow 0)\} \qquad (i/j) = (0/1), (1/2), (0/2). \tag{32}$$

$\tilde{\Gamma}_{(i/j)}$ represents the codimension one phase-transition manifold on which two eigenvalues become degenerated. They meet in the codimension two manifold Γ_* of triply degenerated eigenvalues. As the result of this situation the q-space is divided into three parts (phases) corresponding to the different largest eigenvalue. The scaling relation connected with the manifolds $\tilde{\Gamma}_{(i/j)}$ was treated in section 2. Let us therefore concentrate on Γ_* and choose $q_* \in \Gamma_*$ arbitrary but fixed. $\nu_* := \nu_{q_*}(\delta \downarrow 0)$ denotes the critical eigenvalue. Then by ($\mathrm{P_t}1$) a polynomial of third-order can be extracted from the characteristic equation so that (5) simplifies to

$$0 = \left(\frac{\nu_q}{\nu_*} - 1\right)^3 + a(\delta, q)\left(\frac{\nu_q}{\nu_*} - 1\right)^2 + b(\delta, q)\left(\frac{\nu_q}{\nu_*} - 1\right) + c(\delta, q). \tag{33}$$

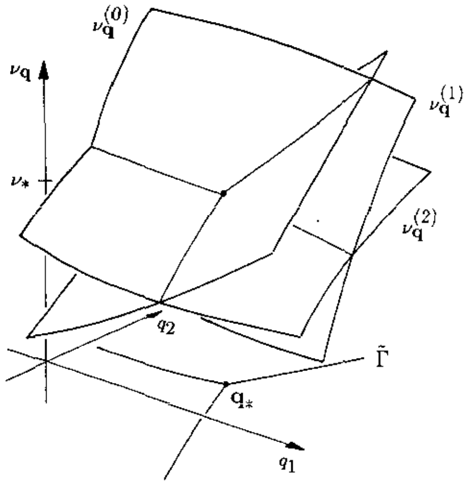

Figure 2. Diagrammatic view of the eigenvalues governing a phase transition based on a triple degeneracy ($\delta \downarrow 0$). $\nu_q^{(0)}$, $\nu_q^{(1)}$, $\nu_q^{(2)}$ denote the three largest eigenvalues, (ν_*, q_*) the codimension two phase-transition point and $\tilde{\Gamma}$ the codimension one phase-transition manifolds in the two-dimensional q-space ($M = 2$).

Instead of this representation the elimination of the second-order term by introducing

$$z := \frac{\nu_q}{\nu_*} - 1 + \tfrac{1}{3}a(\delta, q) \tag{34}$$

turns out to be useful. Then (33) results in

$$z^3 + F(\delta, q)z + G(\delta, q) = 0 \tag{35}$$

where

$$\begin{aligned} F(\delta, q) &= b(\delta, q) - \tfrac{1}{3}a^2(\delta, q) \\ G(\delta, q) &= c(\delta, q) - \tfrac{1}{3}b(\delta, q)a(\delta, q) + \tfrac{2}{27}a^3(\delta, q). \end{aligned} \tag{36}$$

Let us state a few general properties of the quantities (36). If one considers the limit $\delta \downarrow 0$ and takes a q value on the codimension two phase-transition manifold $\tilde{q} \in \Gamma_*$ then (35) admits a triply degenerated solution which means

$$F(\delta \downarrow 0, \tilde{q}) = 0 \qquad G(\delta \downarrow 0, \tilde{q}) = 0 \qquad \tilde{q} \in \Gamma_*. \tag{37}$$

If one chooses especially $\tilde{q} = q_*$ this degenerated eigenvalue coincides with ν_* so that by (33)

$$a(\delta \downarrow 0, q_*) = 0 \qquad b(\delta \downarrow 0, q_*) = 0 \qquad c(\delta \downarrow 0, q_*) = 0. \tag{38}$$

Taking a q value on the neighbouring codimension one manifolds $\tilde{q} \in \tilde{\Gamma}_{(i/j)}$ the limit $\delta \downarrow 0$ yields a doubly degenerated solution of (35) so that

$$D(\delta \downarrow 0, \tilde{q}) = 0 \qquad \tilde{q} \in \tilde{\Gamma}_{(i/j)} \tag{39}$$

where, as an important quantity for the subsequent calculations, the discriminant of (35)

$$D(\delta,q) := \left(\tfrac{1}{3}F(\delta,q)\right)^3 + \left(\tfrac{1}{2}G(\delta,q)\right)^2 \tag{40}$$

has been introduced. Finally we stress that the largest solution of (33) is positive which means that (35) has either three real solutions or one real solution which is larger than the real part of the complex conjugate ones. This constraint results in

$$D(\delta,q) \leqslant 0 \qquad \text{three real solutions} \tag{41a}$$

or

$$D(\delta,q) \geqslant 0 \qquad G(\delta,q) \leqslant 0 \qquad \text{one real solution} \tag{41b}$$

which can be seen, for example, by considering the explicit solutions of the cubic equation (cf (63) and (64)).

Owing to $(\mathrm{P_t}2)$ we are able to write down the following expansions of the coefficients F and G

$$\begin{aligned} F(\delta,q) &= F^{10}\delta + \boldsymbol{F}^{01} : \Delta\boldsymbol{q} : + F^{20}\delta^2 + \boldsymbol{F}^{11} : \Delta\boldsymbol{q} : \delta + \boldsymbol{F}^{02} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : + \mathrm{O}_3 \\ G(\delta,q) &= G^{10}\delta + \boldsymbol{G}^{01} : \Delta\boldsymbol{q} : + G^{20}\delta^2 + \boldsymbol{G}^{11} : \Delta\boldsymbol{q} : \delta + \boldsymbol{G}^{02} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : + G^{30}\delta^3 \\ &\quad + \boldsymbol{G}^{21} : \Delta\boldsymbol{q} : \delta^2 + \boldsymbol{G}^{12} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : \delta + \boldsymbol{G}^{03} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : + \mathrm{O}_4 \end{aligned} \tag{42}$$

where the zero-order contribution vanishes due to (37). By considering special cases we put some constraints on the expansion coefficients.

(i) $\delta \downarrow 0$. Then (42) reads

$$\begin{aligned} F(\delta \downarrow 0,q) &= \boldsymbol{F}^{01} : \Delta\boldsymbol{q} : + \boldsymbol{F}^{02} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : + \mathrm{O}(|\Delta\boldsymbol{q}|^3) \\ G(\delta \downarrow 0,q) &= \boldsymbol{G}^{01} : \Delta\boldsymbol{q} : + \boldsymbol{G}^{02} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : + \boldsymbol{G}^{03} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : + \mathrm{O}(|\Delta\boldsymbol{q}|^4). \end{aligned} \tag{43}$$

By presupposition $(\mathrm{P_t}4)$ (35) possesses three different solutions which can be expanded as $z_i = \boldsymbol{z}_i^1 : \Delta\boldsymbol{q} : + \ldots$, $i = 1,2,3$. Inserting this expansion together with the expression (43) into (35) and requiring three different solutions for the expansion coefficients $\boldsymbol{z}_i^1$ one gets

$$\boldsymbol{F}^{01} = \boldsymbol{0}, \qquad \boldsymbol{G}^{01} = \boldsymbol{0} \qquad \boldsymbol{G}^{02} = \boldsymbol{0}. \tag{44}$$

(ii) $\boldsymbol{q} = \boldsymbol{q}_*$. Now (42) reads

$$\begin{aligned} F(\delta,q_*) &= F^{10}\delta + \mathrm{O}(\delta^2) \\ G(\delta,q_*) &= G^{10}\delta + G^{20}\delta^2 + \mathrm{O}(\delta^3) \end{aligned} \tag{45}$$

and (40) yields

$$D(\delta,q_*) = \left(\tfrac{1}{2}G^{10}\right)^2\delta^2 + \left\{\left(\tfrac{1}{3}F^{10}\right)^3 + \tfrac{1}{2}G^{10}G^{20}\right\}\delta^3 + \mathrm{O}(\delta^4). \tag{46}$$

The relation $G^{10} > 0$ contradicts (41) as in this case from (45) and (46) $G > 0$, $D > 0$. Therefore

$$G^{10} \leqslant 0 \tag{47}$$

holds.

(iii) $\boldsymbol{q}_* = \mathbf{0}$. Because of $(\mathrm{P_t}3)$ $\nu_* = 1$ is valid we conclude from (6) and (33) that

$$c(\delta, \boldsymbol{q} = \mathbf{0}) = 0 \qquad (\boldsymbol{q}_* = \mathbf{0}). \tag{48}$$

By (38) the expansions of $a(\delta, \boldsymbol{q})$ and $b(\delta, \boldsymbol{q})$ contain no contribution of order zero. Inspecting now the relations (36) (48) results in

$$G^{10} = 0 \qquad (\boldsymbol{q}_* = \mathbf{0}). \tag{49}$$

Referring to the discussion following (19) we want to stress that the situation $G^{10} = 0$, $\boldsymbol{q}_* \neq \mathbf{0}$ is not generic. Nevertheless this case will also be treated in the sequel. Its meaning will become clear in the next section. Furthermore we have strong evidence that in the case $\boldsymbol{q}_* = \mathbf{0}$ the Taylor expansion of $b(\delta, \boldsymbol{q}_*)$ contains no contribution of first-order in δ. Even we are not able to show this relation in a strict sense we can give a heuristic explanation in appendix B. As an immediate consequence we obtain from (36)

$$F^{10} = 0 \qquad G^{20} = 0 \qquad (\boldsymbol{q}_* = \mathbf{0}). \tag{50}$$

But then the expansion (42) yields by taking (44) and (49) into account

$$\begin{aligned} G(\delta, \boldsymbol{q}) &= \boldsymbol{G}^{11} : \Delta\boldsymbol{q} : \delta + \mathrm{O}_3 \\ D(\delta, \boldsymbol{q}) &= \left(\tfrac{1}{2}\boldsymbol{G}^{11} : \Delta\boldsymbol{q} : \delta\right)^2 + \mathrm{O}_5. \end{aligned} \tag{51}$$

As $D \geqslant 0$ but G has no definite sign the relation

$$\boldsymbol{G}^{11} = \mathbf{0} \qquad (\boldsymbol{q}_* = \mathbf{0}) \tag{52}$$

follows from the constraint (41).

(iv) $\boldsymbol{q} = \tilde{\boldsymbol{q}} \in \Gamma_*, \delta \downarrow 0$. If one chooses a $\boldsymbol{q}$ value on the codimension two phase-transition manifold (37) yields

$$\begin{aligned} F(\delta \downarrow 0, \tilde{\boldsymbol{q}}) &= \boldsymbol{F}^{02} : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : + \mathrm{O}(|\tilde{\boldsymbol{q}} - \boldsymbol{q}_*|^3) = 0 \\ G(\delta \downarrow 0, \tilde{\boldsymbol{q}}) &= \boldsymbol{G}^{03} : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : + \mathrm{O}(|\tilde{\boldsymbol{q}} - \boldsymbol{q}_*|^4) = 0 \end{aligned} \tag{53}$$

which means

$$\boldsymbol{F}^{02} : \boldsymbol{x} : \boldsymbol{x} := 0 \qquad \boldsymbol{G}^{03} : \boldsymbol{x} : \boldsymbol{x} : \boldsymbol{x} := 0 \qquad \text{if } \boldsymbol{x} \text{ is tangent to } \Gamma_*. \tag{54}$$

(v) $\boldsymbol{q} = \tilde{\boldsymbol{q}} \in \tilde{\Gamma}_{(i/j)}, \delta \downarrow 0$. If the $\boldsymbol{q}$ value is chosen on the codimension one phase-transition manifold we get from (39)

$$\left(\tfrac{1}{3}\boldsymbol{F}^{02} : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* :\right)^3 + \left(\tfrac{1}{2}\boldsymbol{G}^{03} : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* : \tilde{\boldsymbol{q}} - \boldsymbol{q}_* :\right)^2 + \mathrm{O}(|\tilde{\boldsymbol{q}} - \boldsymbol{q}_*|^7) = 0 \tag{55}$$

so that

$$\left(\tfrac{1}{3}F^{02}:x:x:\right)^3+\left(\tfrac{1}{2}G^{03}:x:x:x:\right)^2=0 \qquad \text{if } x \text{ is tangent to } \tilde{\Gamma}_{(i/j)}. \tag{56}$$

By (54) and (56) the coefficients are related to the shape of the phase-transition manifolds. By straightforward algebra the different cases $q_* = 0$ and $q_* \neq 0$ can be discussed separately.

Case A. $q_* = 0$. With respect to (44), (49), (50) and (52) expansion (42) reads as

$$\begin{aligned} F(\delta,q) &= F^{20}\delta^2 + F^{11}:q:\delta + F^{02}:q:q:+\mathrm{O}_3 \\ G(\delta,q) &= G^{30}\delta^3 + G^{21}:q:\delta^2 + G^{12}:q:q:\delta + G^{03}:q:q:q:+\mathrm{O}_4. \end{aligned} \tag{57}$$

Introducing the scaling abbreviations

$$z =: \delta\psi_s \qquad q =: \delta s \tag{58}$$

the eigenvalue equation (35) can be written as

$$\psi_s^3 + F_{\mathrm{A}}(s)\psi_s + G_{\mathrm{A}}(s) + \mathrm{O}(\delta) = 0 \tag{59}$$

where

$$\begin{aligned} F_{\mathrm{A}}(x) &:= F^{20} + F^{11}:x:+F^{02}:x:x: \\ G_{\mathrm{A}}(x) &:= G^{30} + G^{21}:x:+G^{12}:x:x:+G^{03}:x:x:x:. \end{aligned} \tag{60}$$

The discriminant

$$D_{\mathrm{A}}(x) = \left(\tfrac{1}{3}F_{\mathrm{A}}(x)\right)^3 + \left(\tfrac{1}{2}G_{\mathrm{A}}(x)\right)^2 \tag{61}$$

determines whether (59) has entirely real or complex solutions. In the case $D_{\mathrm{A}} < 0$ the three solutions are real which means that by (4) the frequencies vanish. In the opposite case $D_{\mathrm{A}} > 0$ (61) allows for complex solutions that means non-vanishing frequencies. The explicit solutions can be easily written down in both cases by using the formula of Cardano. Rewriting these expressions for the old variables (cf (58) and (34)) one gets the scaling relation

$$\nu_q^{(i)} - 1 = \delta H_{\mathrm{A}}^{(i)}(q/\delta) \qquad i = 0,1,2 \tag{62}$$

where

$$\begin{aligned} & D_{\mathrm{A}}\left(x = \frac{q}{\delta}\right) > 0: \\ & H_{\mathrm{A}}^{(0)}(x) := a_{\mathrm{A}}(x) + 2h_{\mathrm{A}}^{+}(x) \\ & H_{\mathrm{A}}^{(1)}(x) = H_{\mathrm{A}}^{(2)*}(x) := a_{\mathrm{A}}(x) - h_{\mathrm{A}}^{+}(x) + \mathrm{i}\sqrt{3}h_{\mathrm{A}}^{-}(x) \\ & h_{\mathrm{A}}^{\pm}(x) := \tfrac{1}{2}\left(-\frac{G_{\mathrm{A}}(x)}{2}\right)^{1/3}\left\{\left(1+\sqrt{\frac{4D_{\mathrm{A}}(x)}{G_{\mathrm{A}}^2(x)}}\right)^{1/3} \pm \left(1-\sqrt{\frac{4D_{\mathrm{A}}(x)}{G_{\mathrm{A}}^2(x)}}\right)^{1/3}\right\} \\ & a_{\mathrm{A}}(x) := -\tfrac{1}{3}(a^{10} + a^{01}:x:) \end{aligned} \tag{63}$$

and

$$
\begin{aligned}
&D_{\mathrm{A}}\left(x=\frac{q}{\delta}\right)<0: \\
&H_{\mathrm{A}}^{(0)}(x):=a_{\mathrm{A}}(x)+r_{\mathrm{A}}(x)\cos(\Theta_{\mathrm{A}}(x)) \\
&H_{\mathrm{A}}^{(1)}(x):=a_{\mathrm{A}}(x)+r_{\mathrm{A}}(x)\cos\left(\Theta_{\mathrm{A}}(x)+\tfrac{4}{3}\pi\right) \\
&H_{\mathrm{A}}^{(2)}(x):=a_{\mathrm{A}}(x)+r_{\mathrm{A}}(x)\cos\left(\Theta(x)+\tfrac{2}{3}\pi\right) \\
&r_{\mathrm{A}}(x):=2\left|\frac{G_{\mathrm{A}}(x)}{2}\right|^{1/3}\left(1+\frac{4D_{\mathrm{A}}(x)}{G_{\mathrm{A}}^{2}(x)}\right)^{1/6} \\
&\Theta_{\mathrm{A}}(x):=\frac{1}{3}\arg\left(-\frac{G_{\mathrm{A}}(x)}{|G_{\mathrm{A}}(x)|}+\mathrm{i}\sqrt{\left|\frac{4D_{\mathrm{A}}(x)}{G_{\mathrm{A}}^{2}(x)}\right|}\right)\in[0,\pi/3].
\end{aligned}
\tag{64}
$$

It should be stressed that the contribution a_{A} of the scaling functions comes from the expansion of the transformation (34).

Case B. $q_* \neq 0$ ($G^{10}\neq 0$). Using (44) one obtains for the expansion (42)

$$
\begin{aligned}
F(\delta,q)&=F^{10}\delta+F^{20}\delta^{2}+F^{11}:\Delta q:\delta+F^{02}:\Delta q:\Delta q:+\mathrm{O}_{3}\\
G(\delta,q)&=G^{10}\delta+G^{02}\delta^{2}+G^{11}:\Delta q:\delta+G^{30}\delta^{3}+G^{21}:\Delta q:\delta^{2}\\
&\quad+G^{12}:\Delta q:\Delta q:\delta+G^{03}:\Delta q:\Delta q:\Delta q:+\mathrm{O}_{4}.
\end{aligned}
\tag{65}
$$

With the scaling abbreviations

$$
z=:\delta^{1/3}\psi_{s}\qquad \Delta q=:\delta^{1/3}s \tag{66}
$$

the eigenvalue equation (35) reads

$$
\psi_{s}^{3}+F_{\mathrm{B}}(s)\psi_{s}+G_{\mathrm{B}}(s)+\mathrm{O}(\delta^{1/3})=0 \tag{67}
$$

where

$$
\begin{aligned}
&F_{\mathrm{B}}(x):=F^{02}:x:x:\\
&G_{\mathrm{B}}(x):=G^{10}+G^{03}:x:x:x:.
\end{aligned}
\tag{68}
$$

Again the zeros of the discriminant

$$
D_{\mathrm{B}}(x):=\left(\tfrac{1}{3}F_{\mathrm{B}}(x)\right)^{3}+\left(\tfrac{1}{2}G_{\mathrm{B}}(x)\right)^{2} \tag{69}
$$

separate the different regions in q-space where zero and non-vanishing frequencies occur. The scaling relations can be obtained from (67) as in case A and read

$$
\frac{\nu_{q}^{(i)}}{\nu_{*}}-1\simeq\delta^{1/3}H_{\mathrm{B}}^{(i)}\left(\frac{\Delta q}{\delta^{1/3}}\right)\qquad i=0,1,2 \tag{70}
$$

where $H_{\mathrm{B}}^{(i)}$ is given by (63) and (64) with q/δ, D_{A}, G_{A}, a_{A}, $h_{\mathrm{A}}^{\pm}$, r_{A}, Θ_{A} replaced by $\Delta q/\delta^{1/3}$, D_{B}, G_{B}, a_{B}, $h_{\mathrm{B}}^{\pm}$, r_{B}, Θ_{B}. Here a_{B} reads

$$
a_{\mathrm{B}}(x)=-\tfrac{1}{3}a^{01}:x:. \tag{71}
$$

Case C. $q_* \neq 0(G^{10} = 0)$. For later reference we also include this non-generic case in our discussion. The expansion (42) is given by (65) by omitting the term $G^{10}\delta$. Using the scaling abbreviations

$$z =: \sqrt{\delta}\psi_s \qquad \Delta q =: \sqrt{\delta}s \tag{72}$$

the eigenvalue equation (35) yields

$$\psi_s^3 + F_C(s)\psi_s + G_C(s) + O(\sqrt{\delta}) = 0. \tag{73}$$

Here the definitions

$$\begin{aligned} F_C(\boldsymbol{x}) &:= F^{10} + \boldsymbol{F}^{02} : \boldsymbol{x} : \boldsymbol{x} : \\ G_C(\boldsymbol{x}) &:= \boldsymbol{G}^{11} : \boldsymbol{x} : + \boldsymbol{G}^{03} : \boldsymbol{x} : \boldsymbol{x} : \boldsymbol{x} : \end{aligned} \tag{74}$$

have been used and the discriminant

$$D_C(\boldsymbol{x}) = \left(\tfrac{1}{3}F_C(\boldsymbol{x})\right)^3 + \left(\tfrac{1}{2}G_C(\boldsymbol{x})\right)^2 \tag{75}$$

determines the values of the frequencies. The scaling function in this case reads as

$$\frac{\nu_{\boldsymbol{q}}^{(i)}}{\nu_*} - 1 \simeq \sqrt{\delta} H_C^{(i)}\left(\frac{\Delta q}{\sqrt{\delta}}\right) \qquad i = 0, 1, 2 \tag{76}$$

where $H_C^{(i)}$ is again given by (63) and (64) with the obvious substitutions and $a_C(\boldsymbol{x}) := a_B(\boldsymbol{x})$ (cf (71)).

It is clear that the scaling relations for the characteristic functions, the damping rates and the frequencies can be derived immediately from (3), (4), (62), (70) and (76). These scaling relations can be briefly summarized in the equations

$$\begin{aligned} \phi(q) &\simeq \ln \nu_* + \delta^a \bar{\phi}\left(\frac{q - q_*}{\delta^a}\right) \\ \gamma_{\boldsymbol{q}}^{(l)} &\simeq \delta^a \bar{\gamma}^{(l)}\left(\frac{q - q_*}{\delta^a}\right) \qquad l = 1, 2 \\ \omega_{\boldsymbol{q}}^{(l)} &\simeq \delta^a \bar{\omega}^{(l)}\left(\frac{q - q_*}{\delta^a}\right) \qquad l = 1, 2. \end{aligned} \tag{77}$$

Here the value of the exponent $a = 1, \frac{1}{2}, \frac{1}{3}$ depends on the cases already discussed and the scaling functions $\bar{\phi}$, $\bar{\gamma}^{(l)}$ and $\bar{\omega}^{(l)}$ possess two different analytical branches depending on the sign of the discriminant $D(\Delta q/\delta^a)$. Table 2 summarizes the results of this section and gives an overview of the scaling functions. Finally we want to note that (54) and (56) do not allow for a simple separation of tangent and normal variations with respect to the phase-transition manifold. This is different to the case of doubly degenerated eigenvalues where the separation is clearly reflected by (26) and (31).

Table 2. Scaling relations in the case of a triply degenerated eigenvalue and definition of the scaling functions ($X = A, B, C$). For the explicit expressions we refer to ($63_{3,4}$), ($64_{4,5}$), (60), (61), (68), (69), (71), (74) and (75).

	$q_* = 0$	$q_* \neq 0 (G^{10} \neq 0)$	$q_* \neq 0 (G^{10} = 0)$
	$D_A(q/\delta) > 0$	$D_B(\Delta q/\delta^{1/3}) > 0$	$D_C\left(\Delta q/\sqrt{\delta}\right) > 0$
$\phi(q)$	$\delta\bar{\phi}_A^>(q/\delta)$	$\ln\nu_* + \delta^{1/3}\bar{\phi}_B^>(\Delta q/\delta^{1/3})$	$\ln\nu_* + \sqrt{\delta}\bar{\phi}_C^>\left(\Delta q/\sqrt{\delta}\right)$
$\gamma_q^{(1)} = \gamma_q^{(2)}$	$\delta\bar{\gamma}_A^>(q/\delta)$	$\delta^{1/3}\bar{\gamma}_B^>(\Delta q/\delta^{1/3})$	$\sqrt{\delta}\bar{\gamma}_C^>\left(\Delta q/\sqrt{\delta}\right)$
$\omega_q^{(1)} = -\omega_q^{(2)}$	$\delta\bar{\omega}_A^>(q/\delta)$	$\delta^{1/3}\bar{\omega}_B^>(\Delta q/\delta^{1/3})$	$\sqrt{\delta}\bar{\omega}_C^>\left(\Delta q/\sqrt{\delta}\right)$
	$D_A(q/\delta) < 0$	$D_B(\Delta q/\delta^{1/3}) < 0$	$D_C\left(\Delta q/\sqrt{\delta}\right) < 0$
$\phi(q)$	$\delta\bar{\phi}_A^<(q/\delta)$	$\ln\nu_* + \delta^{1/3}\bar{\phi}_B^<(\Delta q/\delta^{1/3})$	$\ln\nu_* + \sqrt{\delta}\bar{\phi}_C^<\left(\Delta q/\sqrt{\delta}\right)$
$\gamma_q^{(1)}$	$\delta\bar{\gamma}_A^{(1)<}(q/\delta)$	$\delta^{1/3}\bar{\gamma}_B^{(1)<}(\Delta q/\delta^{1/3})$	$\sqrt{\delta}\bar{\gamma}_C^{(1)<}\left(\Delta q/\sqrt{\delta}\right)$
$\gamma_q^{(2)}$	$\delta\bar{\gamma}_A^{(2)<}(q/\delta)$	$\delta^{1/3}\bar{\gamma}_B^{(2)<}(\Delta q/\delta^{1/3})$	$\sqrt{\delta}\bar{\gamma}_C^{(2)<}\left(\Delta q/\sqrt{\delta}\right)$
$\omega_q^{(1)} = \omega_q^{(2)}$	0	0	0
$\bar{\phi}_X^>(x) := a_X(x) + h_X^+(x)$		$\bar{\phi}^<(x) := a_X(x) + r_X(x)\cos\Theta_X(x)$	
$\bar{\gamma}_X^>(x) := 3h_X^+(x)$		$\bar{\gamma}_X^{(1)<}(x) := r_X(x)\sin\left(\Theta_X(x) + \frac{2}{3}\pi\right)$	
$\bar{\omega}_X^>(x) := -\sqrt{3}h_X^-(x)$		$\bar{\gamma}_X^{(2)<}(x) := r_X(x)\sin\left(\Theta_X(x) + \frac{1}{3}\pi\right)$	

4. Examples

As shown in the preceeding sections a typical scaling behaviour emerges near the degeneracy points of eigenvalues. Those degeneracies typically occur in the vicinity of bifurcation points, especially when a crisis is involved in the bifurcation. In order to gain some more insight into the three different cases analysed in the preceeding section we want to discuss two simple examples. In general, model systems can be analysed numerically by evaluating the quantity (3). This approach does not require knowledge of the transfer operator and leads to the scaling relations. But it usually requires a large numerical effort and is therefore beyond the scope of this article. Analysing the transfer operator greatly simplifies of the calculation. It is, however, difficult to derive an appropriate expression for the transfer operator of a specific model system. We refer the reader to the literature for the treatment of special examples [18, 19]. In order to avoid this tedious procedure we will concentrate here on simple one-dimensional Markov maps for which the transfer operator can be analysed at least approximately without great effort. Nevertheless our models have some general properties which can also be expected to be valid in more complicated systems.

As a first example let us consider a one-dimensional expanding Markov map which has been derived as a crude approximation for the Lorenz equations† [17] (cf figure 3). Below the bifurcation point the system admits a chaotic invariant set which undergoes an interior crisis as the bifurcation point is reached. The transition matrix, that means the matrix representation of the Frobenius–Perron operator for this map is easily written down. In the vicinity of the bifurcation point this large matrix can be

† In contrast to the original Lorenz equations this map has two unstable fixed points outside the chaotic invariant set in order to yield a chaotic attractor on both sides of the bifurcation point.

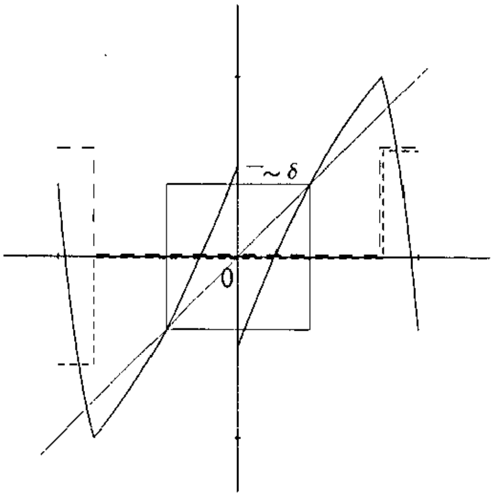

Figure 3. Inversion symmetric Markov map after suffering an interior crisis (solid line). The box indicates the domain of the former attractor and δ denotes the bifurcation parameter. Furthermore the functions $u(x)$ (broken line) and $v(x)$ (dotted line) used for evaluating the characteristic function are shown.

approximated by a transition matrix between the two unstable fixed points and the chaotic repellor [20], an aproximation which also seems to be reasonable from the physical point of view

$$\begin{pmatrix} 1-\alpha & \delta/2 & 0 \\ \alpha & 1-\delta & \alpha \\ 0 & \delta/2 & 1-\alpha \end{pmatrix}. \tag{78}$$

Here α denotes the escape rate from the unstable fixed point and $\delta \ll 1$ the transition rate from the chaotic repellor to one of the fixed points. The dependence of the latter on the bifurcation parameter of the mapping depends on its geometric properties but is linear in the case of Markov maps. The inversion symmetry of the mapping has lead to a symmetry property of the matrix (78). In order to analyse the bifurcation using the quantity (3) a two valued function $\boldsymbol{u}(x) = (u(x), v(x))$ seems to be appropriate where u denotes an even and v an odd function of its argument. The simplest choice is depicted in figure 3, where u and v take the values 1,0,1 respectively -1,0,1 in the neighbourhood of the relevant repellors. Taking the transition matrix (78) into account the matrix representation of the transfer operator then reads as

$$(\mathcal{H}_{\boldsymbol{q}}^{\boldsymbol{u}}) = \begin{pmatrix} (1-\alpha)e^{p-q} & \delta/2 & 0 \\ \alpha e^{p-q} & 1-\delta & \alpha e^{p+q} \\ 0 & \delta/2 & (1-\alpha)e^{p+q} \end{pmatrix} \tag{79}$$

where $\boldsymbol{q} = (p, q)$. It is worth mentioning that the expression (79) which was derived for a very special model system depends only on the symmetry of the system and the type of the bifurcation mentioned earlier. Although the dependence of α and δ on the parameters of the system is based on the special features of the system in a complicated manner it is expected that our simple model shares many of the properties emerging in more complicated ones. At the bifurcation point $\delta \downarrow 0$ the operator (79) possesses a triply degenerated eigenvalue ν_* at the phase-transition point (p_*, q_*)

$$(1-\alpha)\mathrm{e}^{p_*-q_*} = 1 = (1-\alpha)\mathrm{e}^{p_*+q_*} = \nu_* \tag{80}$$

and two linearly independent eigenvectors. Furthermore three phase-transition lines $\{(p,q)\,|\,q-q_* := \Delta q = 0, p-p_* := \Delta p \geqslant 0\}$, $\{(p,q)\,|\,\Delta p = \Delta q \leqslant 0\}$ and $\{(p,q)\,|\,\Delta p = -\Delta q \leqslant 0\}$ meet in this point. These phase-transition lines divide the $\boldsymbol{q}$-plane into three phases which are determined by the two unstable fixed points and the chaotic invariant set respectively. To investigate the scaling behaviour emerging for $\delta > 0$ we analyse the eigenvalue equation of the expression (79). The coefficients of the characteristic equation (35) read in this case

$$\begin{aligned}
F(\delta,\boldsymbol{q}) = & -\tfrac{1}{2}\tilde{\alpha}(\mathrm{e}^{\Delta p+\Delta q}+\mathrm{e}^{\Delta p-\Delta q})\delta - \tfrac{1}{3}(\mathrm{e}^{\Delta p-\Delta q}-\mathrm{e}^{\Delta p+\Delta q})^2 - \tfrac{1}{3}(\mathrm{e}^{\Delta p-\Delta q}-1+\delta) \\
& \times(\mathrm{e}^{\Delta p+\Delta q}-1+\delta) \\
G(\delta,\boldsymbol{q}) = & \tfrac{1}{6}\tilde{\alpha}\{\mathrm{e}^{\Delta p+\Delta q}(\mathrm{e}^{\Delta p-\Delta q}-1+\delta)+\mathrm{e}^{\Delta p-\Delta q}(\mathrm{e}^{\Delta p+\Delta q}-1+\delta) \\
& -(\mathrm{e}^{\Delta p+\Delta q}-\mathrm{e}^{\Delta p-\Delta q})^2\}\delta + \tfrac{2}{3}(\mathrm{e}^{\Delta p-\Delta q}-1) \\
& \times(\mathrm{e}^{\Delta p+\Delta q}-1)\delta - \tfrac{1}{9}\{(\mathrm{e}^{\Delta p-\Delta q}-1)^3-\delta^3+(\mathrm{e}^{\Delta p+\Delta q}-1)^3\} \\
& +\tfrac{1}{27}(\mathrm{e}^{\Delta p-\Delta q}+\mathrm{e}^{\Delta p+\Delta q}-2-\delta)^3
\end{aligned} \tag{81}$$

where

$$\tilde{\alpha} := \alpha/(1-\alpha). \tag{82}$$

Inspecting (81_2) we recognize that $G^{10} = 0$ holds so that case C of section 3 applies. The coefficients entering the scaling functions (74) read

$$\begin{aligned}
&F^{10}\delta = -\tilde{\alpha}\delta \\
&F^{02} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} := -\tfrac{1}{3}(\Delta p)^2 - (\Delta q)^2 \\
&G^{11} : \Delta\boldsymbol{q} : \delta = \tfrac{1}{3}\tilde{\alpha}\Delta p\delta \\
&G^{03} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} : \Delta\boldsymbol{q} := \tfrac{2}{3}\Delta p\left(\frac{(\Delta p)^2}{9} - (\Delta q)^2\right)
\end{aligned} \tag{83}$$

and (74) results in

$$\begin{aligned}
&F_{\mathrm{C}}(s,t) = -\tilde{\alpha} - \tfrac{1}{3}s^2 - t^2 \\
&G_{\mathrm{C}}(s,t) = \tfrac{2}{3}s\left(\frac{\tilde{\alpha}}{2} + \frac{s^2}{9} - t^2\right).
\end{aligned} \tag{84}$$

Here the abbreviations $s_1 = s = \Delta p/\sqrt{\delta}$, $s_2 = t = \Delta q/\sqrt{\delta}$ have been used. The discriminant (75) is, in this case, negative which can be checked by a straightforward computation. As a consequence only real eigenvalues that means vanishing frequencies occur in the vicinity of the phase-transition point and the scaling functions are given by the second part of the third column in table 2.

As shown in the previous example the symmetry of the system has lead to the occurrence of the non-generic case C. In order to discuss the more general case we have to refer back to models which do not share this property. As we are only interested in the principal aspect of this case we will directly investigate a simple matrix representation of the transfer operator although we can also find simple one-dimensional and probably higher dimensional maps from which it can be derived. Let us consider a system which admits one attracting and two repelling sets which decay towards the former one. As a bifurcation parameter is changed a crisis should occur where a transition from the attracting set to one of the repellors should be possible. Then the transition matrix reads as

$$\begin{pmatrix} 1-\delta & \alpha_{21} & \alpha_{31} \\ 0 & 1-\alpha_{21} & \alpha_{32} \\ \delta & 0 & 1-\alpha_{31}-\alpha_{32} \end{pmatrix} . \tag{85}$$

where $\delta \ll 1$ denotes the bifurcation parameter and α_{ij} the decay rates of the repelling sets. If we again consider the observable $\boldsymbol{u}(x) = (u(x), v(x))$ taking constant values $1, 0, 1$ and $-1, 0, 1$ respectively on the attracting and the repelling sets then the transfer operator admits the simple matrix representation

$$(\mathcal{H}_{\boldsymbol{q}}^{\boldsymbol{u}}) = \begin{pmatrix} (1-\delta)e^{p-q} & \alpha_{21} & \alpha_{31}e^{p+q} \\ 0 & 1-\alpha_{21} & \alpha_{32}e^{p+q} \\ \delta e^{p-q} & 0 & (1-\alpha_{31}-\alpha_{32})e^{p+q} \end{pmatrix} . \tag{86}$$

Due to the introduction of two fluctuating quantities we are able to detect the two level internal structure of the repeller. At the bifurcation point $\delta \downarrow 0$ the operator (86) yields a triply degenerated eigenvalue ν_* at the phase-transition point (p_*, q_*)

$$e^{p_*-q_*} = 1-\alpha_{21} = (1-\alpha_{31}-\alpha_{32})e^{p_*+q_*} = \nu_* \tag{87}$$

where the three phase-transition lines $\{(p,q) \mid \Delta p = \Delta q \leqslant 0\}$, $\{(p,q) \mid \Delta p = -\Delta q \leqslant 0\}$ and $\{(p,q) \mid \Delta q = 0, \Delta p \geqslant 0\}$ meet. Furthermore one easily recognizes that the operator admits only one eigenvector which is the generic case. In the case $\delta > 0$ the characteristic equation of the matrix (86) can be obtained after an elementary but exhausting computation. Rewriting it in the form (35) we get for the coefficients

$$\begin{aligned} F(\delta, \boldsymbol{q}) &= -\tilde{\alpha}_{31}e^{2\Delta p}\delta - \tfrac{1}{3}\{(1-\delta)e^{\Delta p-\Delta q} - e^{\Delta p+\Delta q}\}^2 - \tfrac{1}{3}\{(1-\delta)e^{\Delta p-\Delta q} - 1\} \\ &\quad \times \{e^{\Delta p+\Delta q} - 1\} \\ G(\delta, \boldsymbol{q}) &= -\tilde{\alpha}_{21}\tilde{\alpha}_{32}e^{2\Delta p}\delta - \tfrac{1}{3}\tilde{\alpha}_{31}\{(1-\delta)e^{\Delta p-\Delta q} + e^{\Delta p+\Delta q} - 2\}e^{2\Delta p}\delta \\ &\quad - \tfrac{2}{27}\{(1-\delta)e^{\Delta p-\Delta q} - 1\}^3 - \tfrac{2}{27}\{e^{\Delta p+\Delta q} - 1\}^3 \\ &\quad + \tfrac{1}{9}\{(1-\delta)e^{\Delta p-\Delta q} + e^{\Delta p+\Delta q} - 2\}\{(1-\delta)e^{\Delta p-\Delta q} - 1\}\{e^{\Delta p+\Delta q} - 1\} \end{aligned} \tag{88}$$

where the abbreviations

$$\tilde{\alpha}_{21} := \frac{\alpha_{21}}{1-\alpha_{21}} \qquad \tilde{\alpha}_{31} := \frac{\alpha_{31}}{1-\alpha_{31}-\alpha_{32}} \qquad \tilde{\alpha}_{32} := \frac{\alpha_{32}}{1-\alpha_{31}-\alpha_{32}} \tag{89}$$

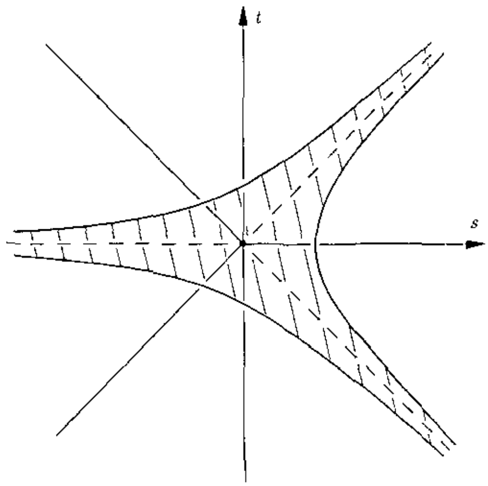

Figure 4. Diagrammatic view of the region in q-space containing finite frequencies in the case $\delta > 0$. The full curves denote the phase-transition lines ($\delta \downarrow 0$) and the broken curves the lines of doubly degenerated eigenvalues ($\delta \downarrow 0$) which can be obtained from (39), (68), (69) and (90).

have been used. Clearly $G^{10} \neq 0$ holds so that case B applies. The coefficients contributing to the scaling functions (68) can be immediately derived. One gets

$$\begin{aligned} F^{02} : \Delta q : \Delta q &:= -\tfrac{1}{3}(\Delta p)^2 - (\Delta q)^2 \\ G^{10}\delta &= -\tilde{\alpha}_{21}\tilde{\alpha}_{32}\delta \\ G^{03} : \Delta q : \Delta q : \Delta q &:= \tfrac{2}{3}\Delta p\left(\frac{(\Delta p)^2}{9} - (\Delta q)^2\right) \end{aligned} \tag{90}$$

and the explicit expressions for the scaling functions read as

$$\begin{aligned} F_{\mathrm{B}}(s,t) &= -\frac{s^2}{3} - t^2 \\ G_{\mathrm{B}}(s,t) &= -\tilde{\alpha}_{21}\tilde{\alpha}_{32} + \tfrac{2}{3}s\left(\frac{s^2}{9} - t^2\right). \end{aligned} \tag{91}$$

Here the abbreviations $s_1 = s = \Delta p/\delta^{1/3}$, $s_2 = t = \Delta q/\delta^{1/3}$ have been introduced. The discriminant (69) is written as

$$D_{\mathrm{B}}(s,t) = -\tfrac{1}{27}\left(\frac{s^2}{3} + t^2\right)^3 + \left(-\frac{\tilde{\alpha}_{21}\tilde{\alpha}_{32}}{2} + \frac{s}{3}\left(\frac{s^2}{9} - t^2\right)\right)^2. \tag{92}$$

Setting this expression equal to zero determines the border line between vanishing and non-vanishing frequencies. It is sketched in figure 4. In the vicinity of the phase-transition point frequencies of order $O(\delta^{1/3})$ occur. Furthermore one recognizes the remarkable fact that far away from the phase-transition point non-vanishing frequencies also emerge in regions of doubly degenerated eigenvalues. This is highly plausible as doubly degenerated real eigenvalues might become complex valued by a small perturbation. On the phase-transition lines, however, this effect is suppressed as the largest eigenvalue has to be a real quantity.

As shown by these examples the three cases A, B, C mentioned in section 3 reflect the number of eigenvectors that the transfer operator admits at the bifurcation point. Therefore the scaling behaviour allows for an investigation of the spectral properties even if an explicit representation for the transfer operator is not known.

5. Conclusion

In this article we have extended a previously developed idea [11] to derive scaling relations in the vicinity of q-phase transition points from a rather general point of view. The generality of our approach is based on avoiding any reference to special dynamical systems but starting directly from the characteristic equation of the transfer operator which determines the eigenvalues of interest. Using a few general properties of this equation we have shown that the relevant thermodynamical quantities obey in general a scaling relation of the form $\delta^{a} H((q-q_*)/\delta^{a})$. Furthermore we have been able to derive explicit expressions for the scaling functions in a mean-field-like way. Besides this the analytical expression for the scaling function depends only on the number of eigenvalues which become degenerated at the bifurcation point and cause the phase transition. The surprising observation that these scaling functions have turned out to be rather independent of the special dynamical system [21] is explained by our results in a clear way. But we note that our approach cannot link the concrete bifurcation parameter of a dynamical system (e.g. an inverse-transition time) to our scaling parameter δ in general. We suppose that such a relation depends on special properties of the dynamical system under consideration [4]. As we have allowed for the treatment of the multivariable case in our framework higher-order degeneracies of eigenvalues can also be achieved in a generic way. The associated phase transitions clearly display the different local structures of the invariant set involved. The treatment of the scaling behaviour in their vicinity has shown that finite frequencies might emerge which mirror the degeneracy of low-lying eigenvalues. Finally we want to stress that our approach is general enough to capture quite different situations which can be described by a transfer operator. Considering, for example, stochastic nonlinear systems it is clear that our approach yields a noise induced scaling behaviour in the vicinity of the phase-transition point even if the precise dependence of the scaling parameter δ on the noise strength is unknown *a priori*. We conclude that phase transitions involving a finite number of eigenvalues are captured by our treatment in a unified way. Nevertheless certain kinds of non-hyperbolic situations, where a continuous part of the spectrum is involved in the phase-transition [9, 10], demands for further investigations.

Acknowledgments

One of the authors (WJ) is greatly indebted to the Japan Society for the Promotion of Science for financial support by the postdoctoral fellowship for foreign researchers and to the Alexander von Humboldt Stiftung for financial support by the Feodor Lynen programme.

Appendix A

Instead of (P_t4) let us assume that the characteristic equation (5) admits one real and two complex conjugated solutions in the vicinity of the phase-transition point ($\delta \downarrow 0$). As we are only interested in the principal aspect of this case we want to restrict the discussion to two variables ($M = 2$) and denote the corresponding parameters by $\boldsymbol{q} = (p, q)$ to avoid unnecessary indices. For reasons that will finally become clear we concentrate on the case $\delta \downarrow 0$ and on the discussion of the phase-transition lines†. The main features are displayed in figure 5 and we show in the sequel that they occur generically.

By the presupposition made above, (5) simplifies in the vicinity of the phase-transition point (p_*, q_*) to

$$\left\{\frac{\nu_{\boldsymbol{q}}}{\nu_*} - 1 - A(\Delta p, \Delta q)\right\}\left\{\left(\frac{\nu_{\boldsymbol{q}}}{\nu_*} - 1\right)^2 + B(\Delta p, \Delta q)\left(\frac{\nu_{\boldsymbol{q}}}{\nu_*} - 1\right) + C(\Delta p, \Delta q)\right\} = 0 \tag{93}$$

if the three largest eigenvalues are considered. Here $\Delta p := p - p_*$, $\Delta q := q - q_*$, $\nu_{\boldsymbol{q}} = \nu_*(1 + A(\Delta p, \Delta q))$ yields the real eigenvalue and the coefficients A, B, C are analytical functions of their arguments. The zero-order term in their expansions vanishes due to (P_t1). Additionally the first-order contribution to $C(\Delta p, \Delta q)$ does not vanish as (93) should yield a complex solution in the neighbourhood of the phase-transition point. Elimination of the quadratic term in $\nu_{\boldsymbol{q}}/\nu_* - 1$ by a linear transformation results in (35) where

$$\begin{aligned} F(\Delta p, \Delta q) &= C - AB - \tfrac{1}{3}(B - A)^2 \\ G(\Delta p, \Delta q) &= -CA - \tfrac{1}{3}(C - AB)(B - A) + \tfrac{2}{27}(B - A)^3. \end{aligned} \tag{94}$$

The arguments of A, B and C have been suppressed to simplify the notation. In contrast to (44) the expansions of F and G contain contributions of the order O_1 and O_2. By (94) both quantities are related via

$$G(\Delta p, \Delta q) = -\tfrac{1}{3}F(\Delta p, \Delta q)\{2A + B\} - \tfrac{1}{27}\{2A + B\}^3. \tag{95}$$

We are interested in the phase-transition lines which are determined by a degenerated eigenvalue. Referring back to (39) these lines are given by the zeros of the discriminant. Using (95) this condition reads

$$0 = D(\Delta p, \Delta q) = \tfrac{4}{27}\left\{F(\Delta p, \Delta q) + \tfrac{1}{3}(2A + B)^2\right\}^2 \times \left\{F(\Delta p, \Delta q) + \tfrac{1}{12}(2A + B)^2\right\}. \tag{96}$$

So we obtain two phase-transition lines in the vicinity of the codimension two phase-transition point which are given by the equations

$$\begin{aligned} \tilde{\Gamma}_1 &= \{(p, q) | F(\Delta p, \Delta q) = -\tfrac{1}{3}(2A + B)^2\} \\ \tilde{\Gamma}_2 &= \{(p, q) | F(\Delta p, \Delta q) = -\tfrac{1}{12}(2A + B)^2\}. \end{aligned} \tag{97}$$

† The argument $\delta \downarrow 0$ will be suppressed in the notation.

These expressions are a trivial consequence of (93). The shape of these lines can easily be obtained using the Taylor expansions of F, A, B. They are qualitatively displayed in figure 5. Inspecting (96) one recognizes that the sign of D does not change if one crosses $\tilde{\Gamma}_1$ ($D \leqslant 0$) whereas it changes by crossing $\tilde{\Gamma}_2$. Referring back to (41) this means that on $\tilde{\Gamma}_1$ two real eigenvalues become degenerate whereas on $\tilde{\Gamma}_2$ a pair of complex conjugated eigenvalues is born. Due to the requirement that the largest eigenvalue is real the last-mentioned transition takes place between the lower lying eigenvalues. Figure 5 summarizes our outcomes.

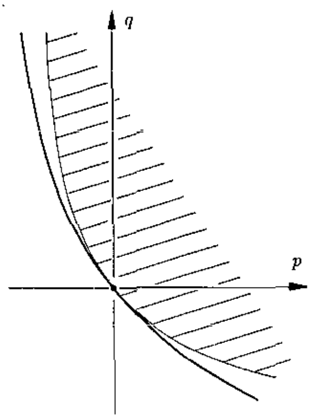

Figure 5. Diagrammatic view of the phase-transition line (thick line) and the line of degeneracy between the lower lying eigenvalues (thin line). In the hatched region the system possesses complex eigenvalues.

It is rather unlikely that the strange looking phase diagram (figure 5) occurs in dynamical systems. On the one hand this kind of phase diagram has not been observed in any dynamical system. On the other hand we have some evidence but no proof at hand that this kind of phase transition is impossible.

Appendix B

Our considerations will mainly be based on the Markov property of the operator (2)

$$\int (\mathcal{H}^{u}_{q=0} h)(x)\mathrm{d}x = \int h(x)\mathrm{d}x \tag{98}$$

which has not yet been used. Suppose that the transfer operator admits some probably infinite dimensional matrix representation at $q = q_* = 0$. It possesses by presupposition three eigenvalues which are well separated from the remaining part of the spectrum. For this reason the matrix representation can be cast into the block form

$$(\mathcal{H}^{u}_{q_*=0}) = \begin{pmatrix} \mathbf{H} & \ddots \\ \mathbf{0} & \ddots \end{pmatrix} \tag{99}$$

where $\mathbf{H}$ denotes a 3×3 matrix which determines the properties of the characteristic equation (33). Concentrating for a moment on the limit $\delta \downarrow 0$ $\mathbf{H}$ possesses the triply degenerated eigenvalue $\nu_* = 1$. Furthermore from (98) and the structure (99) we conclude that the sum of column elements of H equals 1. Both requirements determine H uniquely as the identity matrix. Investigating now the case $\delta > 0$ we remark that the off-diagonal matrix elements have to be of the order $O(\delta)$. Otherwise the presupposition ($P_t 2$) is violated. By this reasoning H takes the form

$$\mathsf{H} = \begin{pmatrix} 1-\delta_{12}-\delta_{13} & \delta_{21} & \delta_{31} \\ \delta_{12} & 1-\delta_{21}-\delta_{23} & \delta_{32} \\ \delta_{13} & \delta_{23} & 1-\delta_{31}-\delta_{32} \end{pmatrix} \tag{100}$$

where $\delta_{ij} = O(\delta)$. The characteristic equation (33) at $q = \mathbf{0}$ is determined by

$$0 = \det \begin{pmatrix} \nu_q - 1 + \delta_{12} + \delta_{13} & -\delta_{21} & -\delta_{31} \\ -\delta_{12} & \nu_q - 1 + \delta_{21} + \delta_{23} & -\delta_{32} \\ -\delta_{13} & -\delta_{23} & \nu_q - 1 + \delta_{31} + \delta_{32} \end{pmatrix}. \tag{101}$$

From this equation it is obvious that $b(\delta, q = \mathbf{0})$ contains no contribution of first-order, that means $b^{10} = 0$.

Appendix C

As mentioned in the introduction the characteristic function $\phi(q)$ is strongly related to the fluctuations of the temporal coarse grained quantity $U_n(x) = \sum_{i=0}^{n-1} u(T^i(x))/n$. We want to make this statement explicit in this appendix.

Let us introduce the distribution function of the fluctuating quantity U_n

$$p_n(\alpha) := \langle \delta(\alpha - U_n(x)) \rangle \sim e^{-n\sigma(\alpha)} \tag{102}$$

where $\alpha = (\alpha_1, \ldots, \alpha_M)$ and the asymptotic behaviour stated on the right-hand side defines the fluctuation spectrum $\sigma(\alpha)$ [12, 22]. In the long time limit U_n approaches its ensemble average $\langle u \rangle$ and p_n tends towards the δ-distribution as $\sigma(\alpha) \geqslant \sigma(\langle u \rangle) = 0$. Clearly $\sigma(\alpha)$ explicitly contains the large fluctuations of U_n. Inserting (102) into the definition (3) we obtain

$$\phi(q) = \lim_{n\to\infty} \frac{1}{n} \ln \int e^{n\alpha q} p_n(\alpha) d\alpha. \tag{103}$$

Using the asymptotic behaviour and evaluating the right-hand side with the saddle point method we end up with the relation [12, 23]

$$\phi(q) = -\min_x \{\sigma(x) - qx\}. \tag{104}$$

This Legendre–Fenchel transform can be inverted easily leading to

$$\sigma(\alpha) = -\min_y \{\phi(y) - \alpha y\}. \tag{105}$$

Assuming the smoothness of ϕ and σ the relations (104) and (105) can be cast into a form more suitable for explicit evaluations. Defining the functions $\bar{\alpha}(q)$ and $\bar{q}(\alpha)$ via the relations

$$q =: \frac{\partial \sigma(x)}{\partial x}\Big|_{x=\bar{\alpha}(q)} \qquad \alpha =: \frac{\partial \phi(y)}{\partial y}\Big|_{y=\bar{q}(\alpha)} \tag{106}$$

the transformation reduces to

$$\phi(q) = q\bar{\alpha}(q) - \sigma(\bar{\alpha}(q)) \qquad \sigma(\alpha) = \alpha\bar{q}(\alpha) - \phi(\bar{q}(\alpha)). \tag{107}$$

The scaling behaviour of ϕ

$$\phi(q) = \ln \nu_* + \delta^a \bar{\phi}\left(\frac{q - q_*}{\delta^a}\right) \tag{108}$$

discovered in sections 2 and 3 carries over to a scaling behaviour for the fluctuation spectrum. From (106) and (108) we get

$$\alpha = \frac{\partial\bar{\phi}(y)}{\partial y}\bigg|_{y=(q-q_*)/\delta^a =: \bar{y}(\alpha)} \tag{109}$$

so that (107) yields the scaling relation

$$\sigma(\alpha) = \alpha q_* - \ln \nu_* + \delta^a\{\alpha\bar{y}(\alpha) - \bar{\phi}(\bar{y}(\alpha))\}. \tag{110}$$

Concerning the explicit expressions of $\bar{\phi}$ and a we refer the reader to tables 1 and 2.

Instead of performing the Legendre–Fenchel transform in (104) and (105) with respect to all variables q and α it is possible to introduce quantities where the transformation is applied only to some of the arguments. The meaning of these quantities can be guessed from the considerations already made. Let us split the observables u and the parameters q into two groups $u = (u^{(1)}, u^{(2)})$ and $q = (q^{(1)}, q^{(2)})$. Having the definition (3) in mind one of the authors introduced a characteristic function with respect to the variables $u^{(1)}$ under the constraint that the variables $U_n^{(2)}$ take fixed values $\alpha^{(2)}$ [12]

$$A(q^{(1)}, \alpha^{(2)}) := \lim_{n\to\infty} \frac{1}{n} \ln\langle \exp(nq^{(1)}U_n^{(1)}(x))\delta(\alpha^{(2)} - U_n^{(2)}(x))\rangle. \tag{111}$$

Here the obvious abbreviation $U_n^{(k)}(x) := \sum_{i=0}^{n-1} u^{(k)}(T^i(x))/n$, $k = 1, 2$ has been used. The function $A(q^{(1)}, \alpha^{(2)})$ thus describes the cross correlation between the local averages $U_n^{(1)}$ and $U_n^{(2)}$. Now we proceed along the lines presented above. Using (102), (111) reduces to

$$A(q^{(1)}, \alpha^{(2)}) = \lim_{n\to\infty} \frac{1}{n} \ln \int e^{nq^{(1)}\alpha^{(1)}} p_n(\alpha^{(1)}, \alpha^{(2)}) d\alpha^{(1)} \tag{112}$$

where, with respect to the previously mentioned separation, the notation $\alpha = (\alpha^{(1)}, \alpha^{(2)})$ has been introduced. The asymptotic behaviour of p_n and the saddle point method allow for an evaluation of the integral with the result

$$A(q^{(1)}, \alpha^{(2)}) = -\min_{x^{(1)}}\{\sigma(x^{(1)}, \alpha^{(2)}) - q^{(1)}x^{(1)}\}. \tag{113}$$

Assuming the smoothness of the functions involved this equation can be further simplified. Introducing the functions $\bar{\alpha}^{(1)}(q^{(1)}, \alpha^{(2)})$ and $\bar{q}^{(2)}(q^{(1)}, \alpha^{(2)})$ via

$$\begin{aligned} q^{(1)} &=: \frac{\partial\sigma(x^{(1)}, \alpha^{(2)})}{\partial x^{(1)}}\bigg|_{x^{(1)}=\bar{\alpha}^{(1)}(q^{(1)},\alpha^{(2)})} \\ \alpha^{(2)} &=: \frac{\partial\phi(q^{(1)}, y^{(2)})}{\partial y^{(2)}}\bigg|_{y^{(2)}=\bar{q}^{(2)}(q^{(1)},\alpha^{(2)})} \end{aligned} \tag{114}$$

we obtain

$$A(q^{(1)},\alpha^{(2)}) = q^{(1)}\bar{\alpha}^{(1)}(q^{(1)},\alpha^{(2)}) - \sigma(\bar{\alpha}^{(1)}(q^{(1)},\alpha^{(2)}),\alpha^{(2)})$$
$$= \phi(q^{(1)},\bar{q}^{(2)}(q^{(1)},\alpha^{(2)})) - \bar{q}^{(2)}(q^{(1)},\alpha^{(2)})\alpha^{(2)}. \tag{115}$$

The last equality can be derived by expressing σ by ϕ via (106) and (107). Applying (108) we finally get the scaling behaviour

$$A(q^{(1)},\alpha^{(2)}) = \ln\nu_* - q_*^{(2)}\alpha^{(2)} + \delta^a\left\{\bar{\phi}\left(\frac{q^{(1)}-q_*^{(1)}}{\delta^a},\bar{y}^{(2)}\left(\frac{q^{(1)}-q_*^{(1)}}{\delta^a},\alpha^{(2)}\right)\right)\right.$$
$$\left. -\bar{y}^{(2)}\left(\frac{q^{(1)}-q_*^{(1)}}{\delta^a},\alpha^{(2)}\right)\alpha^{(2)}\right\} \tag{116}$$

where the function $\bar{y}^{(2)}$ is according to (114) defined by

$$\alpha^{(2)} =: \left.\frac{\partial\bar{\phi}\left((q^{(1)}-q_*^{(1)})/\delta^a, y^{(2)}\right)}{\partial y^{(2)}}\right|_{y^{(2)}=(q^{(2)}-q_*^{(2)})/\delta^a=:\bar{y}^{(2)}\left((q^{(1)}-q_*^{(1)})/\delta^a,\alpha^{(2)}\right)} . \tag{117}$$

References

[1] Ruelle D 1977 *Thermodynamical Formalism* (Reading MA: Addison–Wesley)
Walters P 1982 *An Introduction to Ergodic Theory* (Berlin: Springer)
[2] Bowen R 1975 *Equilibrium States and the Ergodic Theory ofAnosov Diffeomorphisms* (Berlin: Springer)
[3] Fujisaka H 1983 *Prog. Theor. Phys.* **70** 1264
Eckmann J P and Procaccia I 1986 *Phys. Rev.* A **34** 659
Paladin G and Vulpiani A 1987 *Phys. Rep.* **156** 147
Morita T, Hata H, Mori H, Horita T and Tomita K 1988 *Prog. Theor. Phys.* **79** 296
Bessis D, Paladin G, Turchetti G and Vaienti J 1988 *J. Stat. Phys.* **51** 109
Grassberger P, Badii R and Politi A 1988 *J. Stat. Phys.* **51** 135
Paladin G and Vaienti S 1988 *J. Phys. A: Math. Gen.* **21** 4609
[4] Mori H, Hata H, Horita T and Kobayashi T 1989 *Prog. Theor. Phys. Suppl.* **99** 1
[5] Fujisaka H and Inoue M 1987 *Prog. Theor. Phys.* **77** 1334
[6] Fujisaka H 1984 *Prog. Theor. Phys.* **71** 513
Horita T, Hata H, Mori H, Morita T, Tomita K, Kuroki S and Okamoto H 1988 *Prog. Theor. Phys.* **80** 793
[7] Hata H, Horita T, Mori H and Tomita K 1988 *Prog. Theor. Phys.* **80** 809
Tomita K, Hata H, Horita T, Mori H, Morita T, Okamoto H and Tominaga H 1989 *Prog. Theor. Phys.* **81** 1124
Shigematsu H *J. Stat. Phys.* submitted
[8] Hata H, Horita T, Mori H and Morita T 1988 *Prog. Theor. Phys.* **80** 953
Mori N, Kobayashi T, Hata H, Morita T, Horita T and Mori H 1989 *Prog. Theor. Phys.* **81** 60
[9] Feigenbaum M J, Procaccia I and Tel T 1989 *Phys. Rev.* A **39** 5359
[10] Wang X J 1989 *Phys. Rev.* A **40** 6647
[11] Just W and Fujisaka H 1991 *Phys. Lett.* **158A** 385
[12] Fujisaka H and Inoue M 1990 *Phys. Rev.* A **41** 5302
[13] Szepfalusy P and Tel T 1986 *Phys. Rev.* A **34** 2520;
Fujisaka H and Inoue M 1987 *Prog. Theor. Phys.* **78** 268

[14] Fujisaka H and Shibata H 1991 *Prog. Theor. Phys.* **85** 187

[15] Just W 1990 *Phys. Lett.* **150A** 362

[16] Bohr T and Rand D 1989 *Physica* **25D** 387

[17] Guckenheimer J and Holmes P 1986 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Berlin: Springer)

[18] Fujisaka H, Yanagida S and Inoue M 1990 *Nonlinear Dynamics and Particle Acceleration (Tsukuba, Japan) (AIP Conf. Proc. 230)* ed Y H Ichikawa and T Tajima (New York: American Institute of Physics)

[19] Shibata H, Fujisaka H and Mori H *Physica* **A** in press

[20] Just W 1992 *J. Stat. Phys.* **67** 271

[21] Yamaguchi A, Fujisaka H and Inoue M 1989 *Phys. Lett.* **135A** 320
Fujisaka H, Yamaguchi A and Inoue M 1989 *Prog. Theor. Phys.* **81** 1146
Yoshida T, Miyazaki S, Mori H, Kobayashi T, Horita T and Hata H 1989 *Prog. Theor. Phys.* **82** 879
Shibata H, Kuroki S and Mori H 1990 *Prog. Theor. Phys.* **83** 829

[22] Ellis R S 1985 *Entropy, Large Deviations and Statistical Mechanics* (New York: Springer)

[23] Vaienti S 1989 *J. Stat. Phys.* **56** 403